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Exact dynamic stiffness matrix for flexural vibration of three-layered sandwich beams

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Abstract

An exact dynamic member stiffness matrix (exact finite element), which defines the flexural motion of a three-layered sandwich beam with unequal faceplates, is developed from the closed form solution of the governing differential equation. This enables the powerful modelling features associated with the finite element technique to be utilised, including the ability to account for nodal masses, spring support stiffnesses and non-classical boundary conditions. However, such a formulation necessitates the solution of a transcendental eigenvalue problem. This is accomplished using the Wittrick–Williams algorithm, which enables the required natural frequencies to be converged upon to any required accuracy with the certain knowledge that none have been missed. The accuracy of the method is confirmed by comparison with three sets of published results and a final example indicates its range of application.

1. Introduction

The literature dealing with the dynamics of unsymmetric, three-layered sandwich beams is extensive, but does not include a stiffness formulation that accounts in an exact way for the uniform distribution of mass in a member. This is surprising since such a course offers two considerable advantages. The first of these is the opportunity to exploit the powerful modelling features of the stiffness method of analysis. For example, continuous beams with varying member

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properties are easily analysed and it is straightforward to incorporate translational and rotational inertias of nodal masses, spring support stiffnesses and non-classical boundary conditions. The second advantage is that the formulation is exact and results in an idealisation containing the minimum number of elements, while leaving invariant the accuracy to which any particular natural frequency can be calculated. This can be important for higher natural frequencies and should be contrasted with traditional finite elements in which the accuracy is sensitive to the idealisation. However, such a formulation is intractable and necessitates the solution of a transcendental eigenvalue problem. The latter point is resolved herein by adopting the Wittrick–Williams algorithm that enables any required natural frequency to be converged upon to any required accuracy with the certain knowledge that none have been missed.

The sandwich beam considered in this paper comprises two unequal faceplates that are separated by a weaker core layer. The following assumptions are then made: (i) transverse direct strains in the face plates and core are negligible so that small transverse displacements are the same for all points in a normal section; (ii) there is perfect bonding at the core/faceplate interfaces; (iii) the face plates are elastic, isotropic and do not deform in shear; (iv) the linearly elastic core carries only shear and the in-plane normal stresses are assumed to be negligible; (v) the transverse flexural inertia is predominant so that the longitudinal and rotary inertias of the beam may be ignored.

Apart from the fact that damping is not considered, these are the same basic assumptions that were adopted by Kerwin [1], who presented the first vibrational analysis of the problem, and which were also adopted by a number of subsequent authors [2–7]. A comparison of the equations developed in these papers, with the exception of Refs. [6,7], has been given by Mead [8]. In addition, more sophisticated models are available in the literature and could equally well be developed into a stiffness formulation along the lines developed in this paper [9,10].

2. Theory

Figs. 1 and 2 show the positive sense of the forces and displacements experienced by a typical elemental length of a member at some instant during the motion. The equations of horizontal, vertical and moment equilibrium can then be written as

$$(n_1 + n_2)' = 0 \quad \text{or} \quad -n_1' = n_2' = n', \tag{1a,b}$$

$$q' = \mu \ddot{v}, \tag{2}$$

$$q = (m - \bar{m})', \tag{3}$$

respectively, where

$$q = q_1 + q_2 + q_c; \quad m = m_1 + m_2; \quad \mu = \mu_1 + \mu_2 + \mu_c \tag{4a,b,c}$$

and $\bar{m}$ is the couple due to the axial forces n_1 and n_2 that are developed in faceplate 1 (upper) and faceplate 2 (lower), respectively, during bending. As a result of Eq. (1), $\bar{m}$ may be written as

$$\bar{m} = n_1 d = -nd. \tag{5}$$

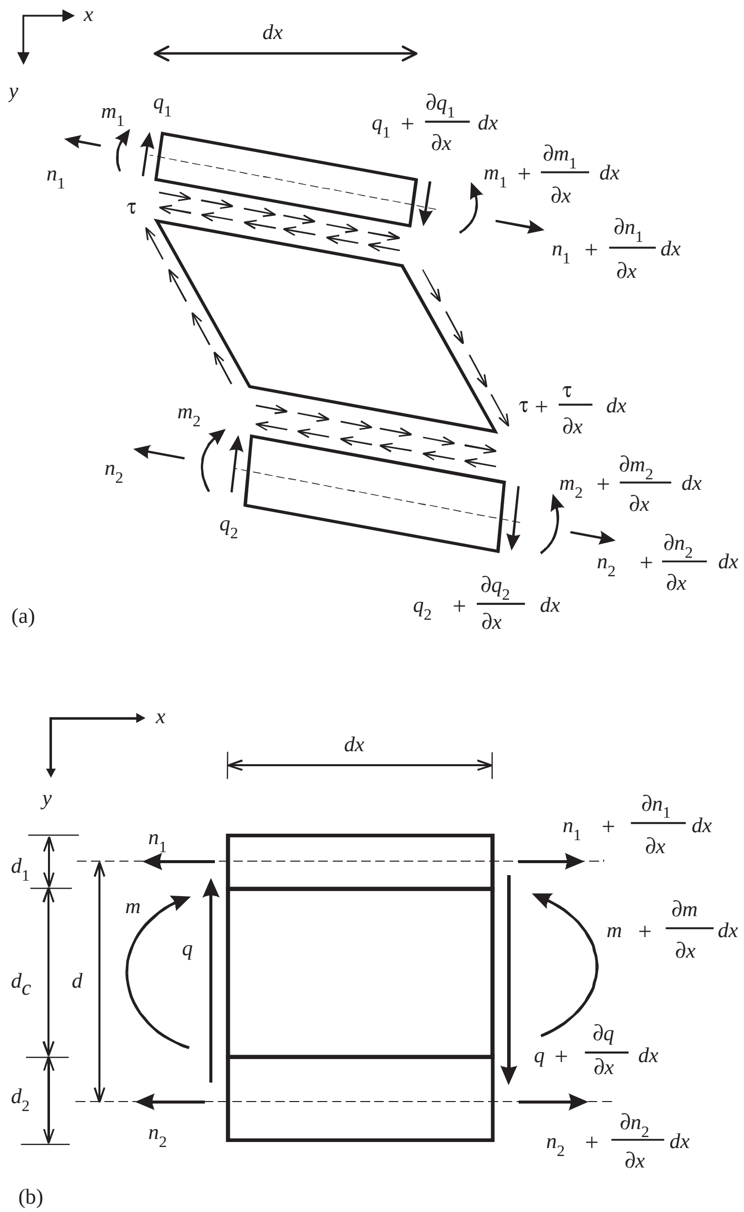

Fig. 1. Positive forces acting on a typical elemental length of a sandwich beam of width b; (a) component member forces and inter member stresses, (b) resultant forces and layer dimensions.

The generic quantities m, q and μ relate to bending moment, shear force and mass/unit length, respectively. When they are un-subscripted, they are resultant or total values and when subscripted with 1, 2, or c they relate to faceplate 1, faceplate 2 and the core, respectively. The prime and dot notations refer to partial differentiation with respect to x and time in the usual way.

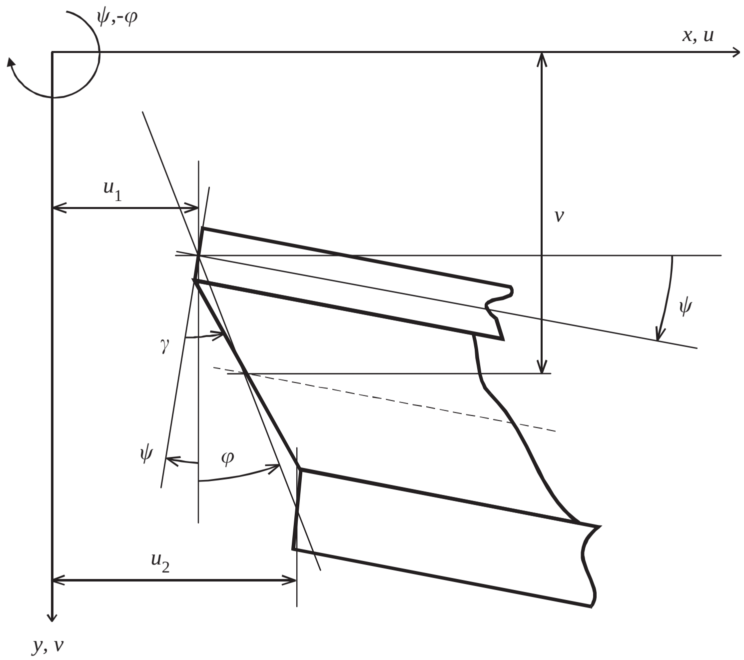

Fig. 2. Coordinate system and positive displacement configuration of a typical section of a sandwich beam of width b.

The necessary force displacement relationships for axial extension and bending of the faceplates are

$$n_i = E_i A_i u_i' \quad \text{and} \quad m_i = -E_i I_i \psi_i' = -E_i I_i v'' \quad (i = 1, 2), \tag{6a,b}$$

respectively, where $E_i A_i$ and $E_i I_i$ are the axial and flexural rigidities of face plate i, and ψ is the slope of the beam's neutral axis.

The core shear stress/strain relationship is easily shown to be

$$\tau/G = d(v' + \phi)/d_c = \gamma \quad \text{where} \quad \phi = (u_2 - u_1)/d, \tag{7a,b}$$

where τ, γ, d_c and G are the shear stress, shear strain, depth and shear modulus related to the core material, respectively, d is the distance between centre lines of the face plates and φ is the average rotation of the beam's cross-section. In addition, stress compatibility at the core face/plate interface requires

$$n_i' = (-1)^i n' = (-1)^i \tau b \quad (i = 1, 2), \tag{8}$$

where b is the breadth of the section. Eqs. (1)–(8) define the motion of the element.

Attention is now confined to harmonic motion in which the time-dependent terms are related to ω, the circular frequency, by

$$f(x, t) = F(x)\mathrm{e}^{\mathrm{i}\omega t} \tag{9}$$

and the upper case characters refer to the amplitude of the equivalent time-dependent quantity. The required equation of motion can then be developed as follows.

From Eqs. (6a), (1b) and (7b) we have

$$U_2' - U_1' = N_2/E_2A_2 - N_1/E_1A_1 = N(1/E_2A_2 + 1/E_1A_1) \tag{10}$$

or

$$N = \varsigma d\Phi' \quad \text{where} \quad \varsigma = E_1A_1E_2A_2/(E_1A_1 + E_2A_2). \tag{11a,b}$$

The first differential of N can also be obtained from Eqs. (7) and (8) as

$$N' = \frac{Gdb}{d_c}(V' + \Phi). \tag{12}$$

Thus, differentiating Eq. (11a) and equating to Eq. (12) yields

$$\Phi'' = \frac{Gb}{\varsigma d_c}(V' + \Phi). \tag{13}$$

Substituting Eqs. (4)–(6) and (12) into (3) gives

$$Q\kappa = \left[\frac{Gd^2b\kappa}{d_c}(V' + \Phi) - V'''\right], \quad \text{where } \kappa = 1/(E_1I_1 + E_2I_2). \tag{14a,b}$$

Substituting Eq. (14) into (2) gives

$$V'''' - \frac{Gd^2b\kappa}{d_c}(V'' + \Phi') - \mu\omega^2\kappa V = 0. \tag{15}$$

Eq. (15) gives Φ' and hence Φ'''. Thus, differentiating Eq. (13) and substituting for Φ' and Φ''' yields the required equation of motion as

$$V'''''' - \frac{Gb}{\varsigma d_c}(1 + \kappa\varsigma d^2)V'''' - \mu\omega^2\kappa\left(V'' - \frac{Gb}{\varsigma d_c}V\right) = 0. \tag{16}$$

Finally, we change to the non-dimensional parameter ξ so that Eq. (16) can be rewritten as

$$[D^6 - \alpha(1 + \beta)D^4 - \lambda(D^2 - \alpha)]V = 0, \tag{17}$$

where

$$\alpha = GbL^2/\varsigma d_c; \quad \beta = \kappa\varsigma d^2; \quad \lambda = \mu\omega^2\kappa L^4; \quad \xi = x/L; \tag{18}$$

D is the operator $\mathrm{d}/\mathrm{d}\xi$ and L is the member length.

The remaining quantities necessary to formulate the required stiffness relationship are readily obtained as follows:

$$\text{Eq. (6b)}: \quad L\Psi = DV, \tag{19}$$

$$\text{Eqs. (13 and 15)}: \quad \alpha^2\beta L\Phi = [D^5 - \alpha\beta D^3 - (\alpha^2\beta + \lambda)D]V, \tag{20}$$

$$\text{Eqs. (13 − 15)}: \quad \alpha\kappa L^3 Q = [D^5 - \alpha(1+\beta)D^3 - \lambda D]V, \tag{21}$$

$$\text{Eqs. (4b) and (6b)}: \quad \kappa L^2 M = -D^2 V, \tag{22}$$

$$\text{Eqs. (5), (11a) and (15)}: \quad \alpha\kappa L^2 \bar{M} = -[D^4 - \alpha\beta D^2 - \lambda]V. \tag{23}$$

Eq. (17) is a linear differential equation with constant coefficients and its solution can be sought in the following form:

$$V = \sum_{j=1}^{6} H_{ij} C_j \zeta_j, \quad \text{where } \zeta_j = \mathrm{e}^{\eta_j \xi} \tag{24a,b}$$

the C_j are arbitrary constants and η_j are the roots of the characteristic equation stemming from Eq. (17). Thus the η_j can be determined as the roots of

$$\eta^6 - \alpha(1+\beta)\eta^4 - \lambda(\eta^2 - \alpha) = 0. \tag{25}$$

The η_j define V, which may be substituted into Eqs. (19)–(23) to yield the following results:

$$\begin{aligned} V &= \sum_{j=1}^{6} H_{1j} C_j \zeta_j, & Q &= \sum_{j=1}^{6} H_{4j} C_j \zeta_j, \\ \Psi &= \sum_{j=1}^{6} H_{2j} C_j \zeta_j, & M &= \sum_{j=1}^{6} H_{5j} C_j \zeta_j, \\ \Phi &= \sum_{j=1}^{6} H_{3j} C_j \zeta_j, & \bar{M} &= \sum_{j=1}^{6} H_{6j} C_j \zeta_j. \end{aligned} \tag{26}$$

Noting that one of the H_{ij} is arbitrary, it is convenient to set $H_{1j} = 1$, which yields the following relationships between the H_{ij} of Eqs. (26)

$$\begin{aligned} H_{1j} &= 1, & H_{4j} &= H_{2j}(H_{5j} - H_{6j}), \\ H_{2j} &= \eta_j / L, & H_{5j} &= -H_{2j}^2/\kappa, \\ H_{3j} &= -H_{2j}(1 + H_{6j}\kappa L^2/\alpha\beta), & H_{6j} &= -(\eta_j^4 - \alpha\beta\eta_j^2 - \lambda)/\alpha\kappa L^2. \end{aligned} \tag{27}$$

The nodal forces and displacements in the local coordinate system shown in Fig. 3(a) are now transformed to the member coordinate system of Fig. 3(b). This is equivalent to imposing the conditions of Eq. (28) onto Eqs. (19)–(23):

$$\begin{aligned} &\text{At } \xi = 0: \quad V = V_1, \quad \Psi = \Psi_1, \quad \Phi = -\Phi_1, \quad Q = -Q_1, \quad M = M_1, \quad \bar{M} = -\bar{M}_1, \\ &\text{At } \xi = 1: \quad V = V_2, \quad \Psi = \Psi_2, \quad \Phi = -\Phi_2, \quad Q = Q_2, \quad M = -M_2, \quad \bar{M} = \bar{M}_2. \end{aligned} \tag{28}$$

The resulting matrix equations are given by

$$\mathbf{d} = \mathbf{SC} \quad \text{and} \quad \mathbf{p} = \mathbf{S}^*\mathbf{C}, \tag{29a,b}$$

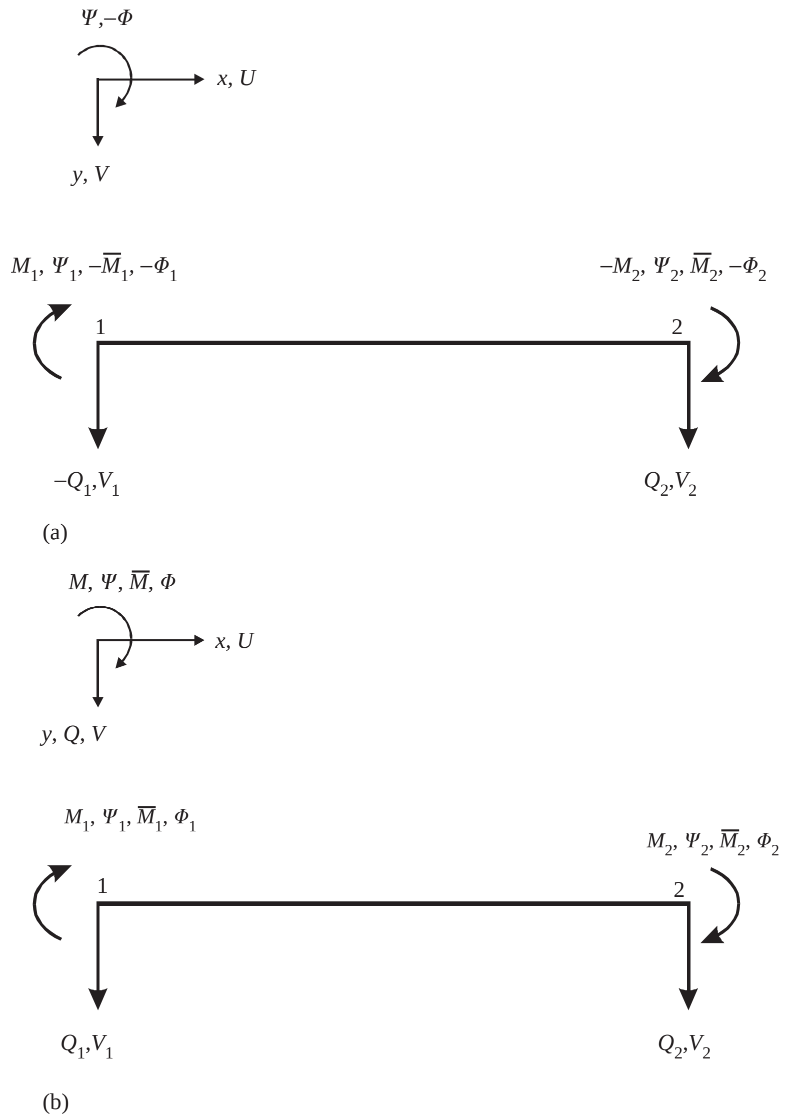

Fig. 3. Nodal forces and displacements (a) in local coordinates, (b) in member coordinates.

where

$$\mathbf{d}=\begin{bmatrix}V_1\\ \Psi_1\\ \Phi_1\\ V_2\\ \Psi_2\\ \Phi_2\end{bmatrix},\quad \mathbf{p}=\begin{bmatrix}Q_1\\ M_1\\ \bar{M}_1\\ Q_2\\ M_2\\ \bar{M}_2\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6\end{bmatrix} \tag{30}$$

and

$$\begin{aligned} &s_{1j} = H_{1j}; \quad s_{2j} = H_{2j}; \quad s_{3j} = -H_{3j}, \\ &s_{4j} = H_{1j}\chi_j; \quad s_{5j} = H_{2j}\chi_j; \quad s_{6j} = -H_{3j}\chi_j, \\ &s^*_{1j} = -H_{4j}; \quad s^*_{2j} = H_{5j}; \quad s^*_{3j} = -H_{6j}, \quad (j = 1, 2, \ldots, 6), \\ &s^*_{4j} = H_{4j}\chi_j; \quad s^*_{5j} = -H_{5j}\chi_j; \quad s^*_{6j} = H_{6j}\chi_j \\ &\chi_j = e^{\eta j}, \end{aligned} \tag{31}$$

where s_{ij} and s^*_{ij} are the elements of $\mathbf{S}$ and $\mathbf{S}^*$, respectively, and their subscripts correspond to row and column coordinates in the usual way. The required dynamic stiffness matrix, $\mathbf{k}$, follows from Eq. (29) through the following steps:

$$\mathbf{C} = \mathbf{S}^{-1}\mathbf{d} \quad \text{therefore} \quad \mathbf{p} = \mathbf{kd}, \quad \text{where } \mathbf{k} = \mathbf{S}^*\mathbf{S}^{-1}. \tag{32}$$

The dynamic stiffness matrix for the overall structure can now be assembled from the element matrices in the usual way. The use of 'exact' finite elements leads to an idealisation containing the minimum number of elements, while leaving invariant the accuracy to which any particular natural frequency can be converged upon. This can be important for higher natural frequencies and should be contrasted with traditional finite elements in which the accuracy is sensitive to the idealisation. Once the required natural frequencies have been determined, the corresponding mode shapes can be retrieved by any reliable method, such as described in Ref. [11]. The method for converging with certainty on the required natural frequencies is now described.

3. Wittrick–Williams algorithm

The dynamic structure stiffness matrix, $\mathbf{K}$, when assembled from the element matrices, yields the required natural frequencies as solutions of the equation

$$\mathbf{KD} = \mathbf{0}, \tag{33}$$

where $\mathbf{D}$ is the vector of amplitudes of the harmonically varying nodal displacements and $\mathbf{K}$ is a function of ω, the circular frequency. In most cases the required natural frequencies correspond to $|\mathbf{K}|$, the determinant of $\mathbf{K}$, being equal to zero. Traditionally the required values have been ascertained by merely tracking the value of $|\mathbf{K}|$ and noting the value of ω corresponding to $|\mathbf{K}| = 0$. However, when $\mathbf{K}$ is developed from exact member theory the determinant is a highly irregular, transcendental function of ω. Additionally, several natural frequencies may be close together or coincident, while others may exceptionally correspond to $\mathbf{D} = \mathbf{0}$. Thus any trial and error method which involves computing $|\mathbf{K}|$ and noting when it changes sign through zero, can miss roots. This danger can be completely overcome by use of the Wittrick–Williams algorithm [12], which has received wide attention [13,14]. The algorithm states that

$$J = J_0 + s\{\mathbf{K}\}, \tag{34}$$

where J is the number of natural frequencies of the structure exceeded by some trial frequency, ω^*, J_0 is the number of natural frequencies which would still be exceeded if all the elements were clamped at their ends so as to make $\mathbf{D} = \mathbf{0}$, and $s\{\mathbf{K}\}$ is the sign count of the matrix $\mathbf{K}$. $s\{\mathbf{K}\}$ is defined in Ref. [12] and is equal to the number of negative elements on the leading diagonal of the upper triangular matrix obtained from $\mathbf{K}$, when $\omega = \omega^*$, by the standard form of Gauss elimination without row interchanges.

The knowledge of J corresponding to any trial frequency makes it possible to develop a method for converging upon any required natural frequency to any desired accuracy. However, while $s\{\mathbf{K}\}$ is easily computed, the value of J_0 is more difficult to determine and is dealt with below.

4. Determination of J_0

From the definition of J_0 it can be seen that

$$J_0 = \sum J_m, \tag{35}$$

where J_m is the number of natural frequencies of a component element, with its ends clamped, which have been exceeded by ω^*, and the summation extends over all such elements. In some cases it is possible to determine the value of J_m for the element type symbolically, using a direct approach [11]. However, this is impractical in the present case due to the algebraic complexity of the expressions. Instead, the same result is achieved by an argument based on Eq. (34) which was originally put forward by Howson and Williams [15].

Consider an element, which has been isolated from the remainder of the structure by clamping its ends. Unfortunately, this structure cannot be solved easily. We therefore seek to establish a different set of boundary conditions that admit a simple symbolic solution and which enable solutions to the clamped ended case to be deduced. This is often most easily achieved by imposing simple supports which, in this case, permit rotation and relative motion of the faceplates, i.e. Ψ and Φ, respectively, but prevent lateral displacement V.

Let the stiffness matrix for this structure be $\mathbf{k}^{ss}$, then the number of roots exceeded by ω^* is given by Eq. (34) and the arguments above as

$$J_{ss} = J_m + s\{\mathbf{k}^{ss}\}, \tag{36}$$

where J_{ss} is the number of natural frequencies that lie below the trial frequency for the element with simple supports. It then follows directly that

$$J_m = J_{ss} - s\{\mathbf{k}^{ss}\}. \tag{37}$$

Once more $\mathbf{k}^{ss}$, and hence $s\{\mathbf{k}^{ss}\}$, is readily obtained, this time from Eq. (32). J_{ss} is slightly more difficult, but relates to the element with boundary conditions that yield a simple exact solution, as shown below.

For the simply supported case, the boundary conditions are defined by

$$M = \bar{M} = V = 0. \tag{38}$$

These conditions are satisfied by assuming solutions of the form

$$V = B \sin n\pi\xi \quad (n = 1, 2, 3, \ldots) \tag{39}$$

and B is a constant. Substituting Eq. (39) and its derivatives into Eq. (17) yields the equation of motion as

$$[n^6\pi^6 + \alpha(1+\beta)n^4\pi^4 - \lambda(n^2\pi^2 + \alpha)]B = 0, \tag{40}$$

which, for non-trivial solutions yields

$$\omega_n = n^2\pi^2\{[n^2\pi^2 + \alpha(1+\beta)]/\mu\kappa L^4(n^2\pi^2+\alpha)\}^{1/2}. \tag{41}$$

Hence J_{ss} is given by the number of positive values of ω_n that lie below the trial frequency, ω^*.
Thus, substituting Eq. (37) into Eq. (35) gives

$$J_0 = \sum (J_{ss} - s\{\mathbf{k}^{ss}\}). \tag{42}$$

The required value of J then follows from Eq. (34).

5. Numerical results

Four examples are now given to validate the theory and indicate its range of application. The first three examples compare results obtained by a number of authors for a simply supported, cantilevered and fixed ended beam, respectively, which have been widely used as test examples. The final example gives only the authors' results for a simple three span continuous beam with various combinations of parameters and support conditions. It can also be used to demonstrate how the conventional method of determinant tracking to find natural frequencies can miss roots.

Example 1. A simply supported sandwich beam with identical faceplates and the following material and geometric properties is analysed. The results obtained by a number of authors are given in Table 1.

$E_1 = E_2 = 68.9\,\text{GPa}$, $G = 82.68\,\text{MPa}$, $\mu_i = \rho_i b d_i$, where $i = 1, 2, c$ and ρ denotes material density, $\rho_1 = \rho_2 = 2680\,\text{kg/m}^3$, $\rho_c = 32.8\,\text{kg/m}^3$, $d_1 = d_2 = 0.4572\,\text{mm}$, $d_c = 12.7\,\text{mm}$, $b = 25.4\,\text{mm}$, $L = 0.9144\,\text{m}$.

Table 1
Comparative results for the first ten natural frequencies (Hz) of the simply supported sandwich beam of Example 1

Freq. no.	Present authors	Rao [6]	Mead [16][a]		Ahmed [17]	Ahmed [18]	Sakiyama [7]	Kameswara [19]	Marur [20]
			Eq. (32)	Table I					
1	57.1358	57.1358	57.1352	56.028	55.5	57.5	56.159	57.068	57.041
2	219.585	219.585	219.575	—	—	—	215.82	218.569	218.361
3	465.172	465.172	465.129	457.12	451	467	457.22	460.925	460.754
4	768.177	768.177	768.058	—	—	—	755.05	757.642	758.692
5	1106.68	1106.68	1106.43	1090.26	1073	1111	1087.9	1086.955	1097.055
6	1465.10	1465.10	1464.65	—	—	—	1440.3	1433.920	1457.064
7	1833.55	1833.55	1832.82	1809.8	1779	1842	1802.7	1789.345	1849.380
8	2206.19	2206.19	2205.09	—	—	—	2169.8	2147.969	2275.916
9	2579.79	2579.79	2578.22	2549.5	2510	2594	2538.2	—	2562
10	2952.65	2952.65	2950.52	—	—	—	2906.2	—	—

[a]Column five contains the results that were presented by Mead in Table I of Ref. [16] and which were calculated using Eq. (32) of the same paper. Unfortunately, Mead provided insufficient data to confirm these results. Thus Eq. (32) was used to calculate the results presented in column four from the data used to determine the results in columns two and three.

Example 2. The beam of Example 1 is now constrained to act as a cantilever and its length is reduced to 0.7112 m. Results for the first eight natural frequencies are presented in Table 2.

Example 3. A fixed ended sandwich beam with the following material and geometrical properties is now considered. The results are shown in Table 3.

$E_1 = E_2 = 68.9\,\text{GPa}$, $G = 68.9\,\text{MPa}$, $\rho_1 = \rho_2 = 2687.3\,\text{kg/m}^3$, $\rho_c = 119.69\,\text{kg/m}^3$, $d_1 = d_2 = 0.40624\,\text{mm}$, $d_c = 6.3475\,\text{mm}$, $b = 25.4\,\text{mm}$, $L = 1.21872\,\text{m}$.

Example 4. Attention is now given to the three span continuous sandwich beam of Fig. 4, for which two sets of results are shown in Table 4. Type A results, in which the basic material and geometric properties of each span are identical to the beam of Example 3 and Type B results, which use the same data, except that the top faceplate in the middle span has the following properties: $E_1 = 207\,\text{GPa}$, $\rho_1 = 7850\,\text{kg/m}^3$, $d_1 = 0.4572\,\text{mm}$. In addition, various combinations of nodal mass and spring support stiffness are imposed as indicated. The rotational inertia $I_{M_1^*}$ of mass M_1^* is assumed to be significant and dependent upon Ψ alone whenever it is included. In similar fashion, K_3 is applied to Φ alone when determining the results in the last three columns of

Table 2
Comparative results for the first eight natural frequencies (Hz) of the cantilevered sandwich beam of Example 2

Freq. no.	Present authors	Ahmed [17]	Ahmed [18]	Sakiyama [7]	Marur [20]
1	33.7513	32.79	33.97	33.146	33.7
2	198.992	193.5	200.5	195.96	197.5
3	512.307	499	517	503.43	505.5
4	907.299	886	918	893.28	890.5
5	1349.65	1320	1368	1328.5	1321
6	1815.82	1779	1844	1790.7	1786
7	2292.45	2249	2331	2260.2	2271
8	2772.23	2723	2824	2738.9	2792

Table 3
Comparative results for the first ten natural frequencies (Hz) of the fixed ended sandwich beam of Example 3. It should be noted that the results presented by Raville are experimental

Freq. no.	Present authors	Sakiyama [7]	Raville [21]
1	34.5965	33.563	—
2	93.1000	90.364	—
3	177.155	172.07	185.5
4	282.784	274.91	280.3
5	406.325	395.42	399.4
6	544.331	530.34	535.2
7	693.787	676.85	680.7
8	852.153	832.43	867.2
9	1017.35	995.36	1020
10	1187.70	1163.9	1201

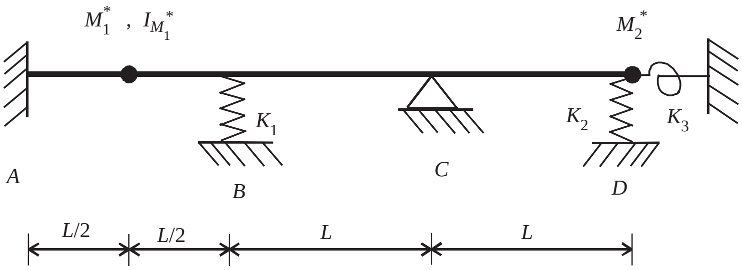

Fig. 4. Three-span continuous sandwich beam of Example 4.

Table 4
The first ten natural frequencies (Hz) of the sandwich beam described in Example 4 for various combinations of nodal mass and stiffness

M_1^* (kg)	0	0	10	10	0	0	0	10	10
$I_{M_1^*}$ (kg m^2)	0	0	0	0.0125	0	0	0	0.0125	0.0125
M_2^* (kg)	0	0	0	0	10	0	0	10	10
K_1 (N/m)	10^{10} (∞)	0	0	0	0	10^7	0	10^7	10^7
K_2 (N/m)	10^{10} (∞)	0	0	0	0	0	0	0	0
K_3 (N m/rad)	10^{10} (∞)	0	0	0	0	0	10^2	10^2	10^2
Freq. No.	Type A								Type B
1	19.8054	3.00131	0.99164	0.98953	0.16468	3.74279	4.68781	0.34959	0.37080
2	28.7227	7.96371	3.36863	3.36577	7.26450	19.9672	8.57050	1.73349	1.78143
3	34.5965	20.6721	11.8279	7.15669	17.6599	28.6420	21.4203	7.40576	7.4113
4	69.3442	29.6883	27.1897	12.1672	24.0689	34.5035	35.5282	20.3410	18.4876
5	84.0878	44.4183	37.1184	27.3358	43.3952	69.4838	45.6496	29.5450	28.4164
6	93.1000	70.7205	67.7937	37.5765	65.2770	84.5535	71.8501	65.8958	60.7004
7	146.067	86.6974	86.5840	68.6572	76.5809	93.5183	94.5076	79.1142	79.1600
8	165.676	109.605	106.240	86.9180	108.084	146.492	110.979	92.9833	91.5582
9	177.155	148.162	127.561	108.150	140.591	166.838	149.409	122.165	121.186
10	246.735	170.175	165.198	127.579	155.788	178.217	179.271	145.313	136.490

Table 4, but applied to both Ψ and Φ when modelling the fixed end condition for the results in the second column.

6. Discussion

The results presented in Tables 1–3 show good correlation between those of the current theory and a selection of comparable results available in the literature. The differences in the results are attributable to many factors that vary widely from approximate solution techniques to differences in basic assumptions. However, a complete description of these differences is deemed to be beyond the scope of the current paper. Finally, Table 4 provides a range of 'exact' solutions, which may be helpful for future comparisons. It can also be used in conjunction with Table 3 to illustrate a possible pit fall that exists if the natural frequencies of a transcendental eigenvalue problem are acquired using a determinant search technique, as follows.

Consider the structure of Fig. 4 with $M_1^* = M_2^* = I_{M_1^*} = 0$, $K_1 = K_2 = K_3 = \infty$ and beams with identical material and geometric properties in each span that correspond to the beam of Example 3. The resulting structure is a uniform beam that is continuous across the simple supports at B and C, clamped at A and D and symmetric about its mid point. The structure could therefore be modelled using one element per member and two nodes, namely B and C.

Since the structure is symmetric, the modes of vibration must be either symmetric or antisymmetric about the midpoint. Considering only the symmetric modes, it is clear that they fall into one of two categories. Either there is rotation at B and C or there is not. For the former category and all the antisymmetric modes, the requirement for a natural frequency, described previously by Eq. (33) as $\mathbf{KD}=\mathbf{0}$, is satisfied in the usual way by $|\mathbf{K}| = 0$. However, frequencies in the latter category satisfy $\mathbf{KD}=\mathbf{0}$ by virtue of the fact that $\mathbf{D}=\mathbf{0}$ rather than the more usual $|\mathbf{K}| = 0$. In fact, $|\mathbf{K}|$ becomes infinite at such natural frequencies, with the consequence that they could be missed by traditional methods of determinant tracking which seek only $|\mathbf{K}| = 0$. Moreover, even if an analyst were to intervene in what is likely to be an automated process, the occurrence of $|\mathbf{K}|$ becoming infinite would not necessarily alert him to the danger that natural frequencies were being missed, since it is quite common for $|\mathbf{K}|$ to change sign through infinity at frequencies which do not correspond to natural frequencies of the structure. The fact that such a condition can arise in simple, practical structures can be seen by comparing the second columns of Tables 3 and 4. This shows that the third, sixth and ninth natural frequencies of the continuous beam of Example 4 correspond to the first three natural frequencies of the fixed ended beam of Example 3, i.e. the clamped ended frequencies of each span member.

The use of the stiffness method offers great flexibility to impose 'constraints' on any selected node. These will typically take the form of mass inertias, spring support stiffnesses or relationships that constrain one or more displacements to move in a predefined way relative to another set of displacements. Imposing such constraints follows the normal rules that would apply to a traditional beam element, except more care is required to associate the constraint with the appropriate degree(s) of freedom. Table 5 amplifies this by listing the possible rotational displacement constraints and how they are achieved, primarily as an aid to establishing boundary conditions.

Table 5
Rotational displacement constraints. A(B) implies an infinitely stiff rotational spring imposed on $\Phi(\Psi)$

Degree of freedom		Constraint	Description
$\Psi \neq 0$	$\Phi \neq 0$	$\Psi = \Phi$	Riveted
$\Psi \neq 0$	$\Phi \neq 0$	None	Free/Free
$\Psi \neq 0$	$\Phi = 0$	A	Free/Fixed
$\Psi = 0$	$\Phi \neq 0$	B	Fixed/Free
$\Psi = 0$	$\Phi = 0$	A + B	Fixed/Fixed

Finally, it is worth noting that while assembling the results it became apparent that it is relatively easy to generate an example in which the value of α, defined in Eq. (18), is sufficiently large to cause the roots of the characteristic equation, Eq. (25), to become sufficiently large that the value of ζ in Eq. (24) overflows, even when using double precision arithmetic. However, it is easy to show that the dominant term in the expression for α is the length of the member (element). Thus if difficulty is experienced it is merely necessary to subdivide the member into a greater number of elements until the problem is resolved.

7. Conclusions

A method for converging with certainty upon any required natural frequency of a single or continuous sandwich beam has been presented. It uses exact member theory in conjunction with the dynamic stiffness technique and this necessitates the solution of a transcendental eigenvalue problem. Solutions are achieved by use of the Wittrick–Williams algorithm, which yields the required natural frequencies to any desired accuracy in such a way that no difficulties are experienced with close or coincident natural frequencies or those exceptional natural frequencies which correspond to the nodal displacement vector being zero. The method therefore provides a very attractive alternative to the traditional finite element technique in which the accuracy is sensitive to the idealisation.

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